

# $n$ -ary Hom-Nambu algebras

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## Abstract

In this paper, we define  $\omega$ -derivations, and study some properties of  $\omega$ -derivations, with its properties we can structure a new  $n$ -ary Hom-Nambu algebra from an  $n$ -ary Hom-Nambu algebra. In addition, we also give derivations and representations of  $n$ -ary Hom-Nambu algebras.

*Key words:*  $n$ -ary Hom-Nambu algebras,  $\omega$ -derivations, representations

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## 1 Introduction

Leibniz  $n$ -algebras were introduced by Casas, J. M. in [2],  $n$ -ary Hom-Nambu algebra is a generalization of the Leibniz  $n$ -algebra. An  $n$ -ary Hom-Nambu algebra is a triple  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$  consisting of a vector space  $\mathfrak{g}$ , a multilinear map  $[\cdot, \dots, \cdot] : \underbrace{\mathfrak{g} \times \dots \times \mathfrak{g}}_n \rightarrow \mathfrak{g}$

and a family  $\alpha = (\alpha_i)_{1 \leq i \leq n-1}$  of linear maps  $\alpha_i : \mathfrak{g} \rightarrow \mathfrak{g}$ , satisfying

$$\begin{aligned} & [[x_1, \dots, x_n], \alpha_1(y_1), \dots, \alpha_{n-1}(y_{n-1})] \\ &= \sum_{i=1}^n [\alpha_1(x_1), \dots, \alpha_{i-1}(x_{i-1}), [x_i, y_1, \dots, y_{n-1}], \alpha_i(x_{i+1}), \dots, \alpha_{n-1}(x_n)]. \end{aligned}$$

When  $\alpha_i = \text{id}$ , it becomes a Leibniz  $n$ -algebra. For  $n = 2$ , one recovers Hom-Leibniz algebras, the specific content about Hom-Leibniz algebras can be seen in [1,3].

In this paper, the main result is to study  $\omega$ -derivations of a vector space, and how to structure a new  $n$ -ary Hom-Nambu algebra. In addition, we also introduce the derivations

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and representations similar with  $n$ -ary multiplicative Hom-Nambu-Lie superalgebras and  $n$ -Lie superalgebras in [4,6] as its application.

The paper is organized as follows. In section 2, we give the definition of  $n$ -ary Hom-Nambu algebra and some examples. The definition of derivation is extended to  $n$ -ary Hom-Nambu algebras from Hom-Lie algebras, and all derivations structure a Hom-Lie algebra in section 3. The most important part is section 4, we define  $\omega$ -derivations, and study some properties of  $\omega$ -derivations. We can structure a new  $n$ -ary Hom-Nambu algebra from a  $(kn + 1)$ -ary Hom-Nambu algebra with these properties. In section 5, we also give representations of  $n$ -ary Hom-Nambu algebras.

## 2 $n$ -ary Hom-Nambu algebra

**Definition 2.1.** [2] *A Leibniz  $n$ -algebra is a pair  $(\mathfrak{g}, [\cdot, \dots, \cdot])$  consisting of a vector space  $\mathfrak{g}$  and a multilinear map  $[\cdot, \dots, \cdot] : \underbrace{\mathfrak{g} \times \dots \times \mathfrak{g}}_n \rightarrow \mathfrak{g}$ , satisfying*

$$\begin{aligned} & [[x_1, \dots, x_n], y_1, \dots, y_{n-1}] \\ &= \sum_{i=1}^n [x_1, \dots, x_{i-1}, [x_i, y_1, \dots, y_{n-1}], x_{i+1}, \dots, x_n]. \end{aligned}$$

**Definition 2.2.** [7] *An  $n$ -ary Hom-Nambu algebra is a triple  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$  consisting of a vector space  $\mathfrak{g}$ , a multilinear map  $[\cdot, \dots, \cdot] : \underbrace{\mathfrak{g} \times \dots \times \mathfrak{g}}_n \rightarrow \mathfrak{g}$  and a family  $\alpha = (\alpha_i)_{1 \leq i \leq n-1}$  of linear maps  $\alpha_i : \mathfrak{g} \rightarrow \mathfrak{g}$ , satisfying*

$$\begin{aligned} & [[x_1, \dots, x_n], \alpha_1(y_1), \dots, \alpha_{n-1}(y_{n-1})] \\ &= \sum_{i=1}^n [\alpha_1(x_1), \dots, \alpha_{i-1}(x_{i-1}), [x_i, y_1, \dots, y_{n-1}], \alpha_i(x_{i+1}), \dots, \alpha_{n-1}(x_n)]. \end{aligned} \quad (2.1)$$

*An  $n$ -ary Hom-Nambu algebra  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$  is multiplicative, if  $\alpha = (\alpha_i)_{1 \leq i \leq n-1}$  with  $\alpha_1 = \dots = \alpha_{n-1} = \alpha$  and satisfying*

$$\alpha([x_1, \dots, x_n]) = [\alpha(x_1), \dots, \alpha(x_n)], \forall x_1, x_2, \dots, x_n \in \mathfrak{g}.$$

If the  $n$ -ary Hom-Nambu algebra  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$  is multiplicative, then the equation (2.1) can be read:

$$\begin{aligned} & [[x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})] \\ &= \sum_{i=1}^n [\alpha(x_1), \dots, \alpha(x_{i-1}), [x_i, y_1, \dots, y_{n-1}], \alpha(x_{i+1}), \dots, \alpha(x_n)]. \end{aligned} \quad (2.2)$$

**Definition 2.3.** [5] A Hom-Lie algebra is a triple  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha)$  consisting of a vector space  $\mathfrak{g}$ , a bilinear map (bracket)  $[\cdot, \cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  and a map  $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying

$$[x, y] = -[y, x],$$

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \quad \forall x, y, z \in \mathfrak{g}.$$

**Definition 2.4.** [5] Let  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  be a Hom-Lie algebra. Then  $I$  is a Hom-ideal of  $\mathfrak{g}$ , if  $I$  satisfies  $[I, \mathfrak{g}] \subseteq I$  and  $\alpha(I) \subseteq I$ .

**Definition 2.5.** Let  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$  and  $(\mathfrak{g}', [\cdot, \dots, \cdot]', \alpha')$  be two  $n$ -ary Hom-Nambu algebras, where  $\alpha = (\alpha_i)_{1 \leq i \leq n-1}$  and  $\alpha' = (\alpha'_i)_{1 \leq i \leq n-1}$ . A linear map  $f : \mathfrak{g} \rightarrow \mathfrak{g}$  is an  $n$ -ary Hom-Nambu algebra morphism if it satisfies

$$f[x_1, \dots, x_n] = [f(x_1), \dots, f(x_n)]',$$

$$f \circ \alpha_i = \alpha'_i \circ f, \forall i = 1, \dots, n-1.$$

**Example 2.6.** Let  $(\mathfrak{g}, [\cdot, \dots, \cdot])$  be a Leibniz  $n$ -algebra and let  $\rho : \mathfrak{g} \rightarrow \mathfrak{g}$  be a Leibniz  $n$ -algebra morphism. Then  $(\mathfrak{g}, \rho \circ [\cdot, \dots, \cdot], \rho)$  is a multiplicative  $n$ -ary Hom-Nambu algebra.

*Proof.* Put  $[\cdot, \dots, \cdot]_{\rho} := \rho \circ [\cdot, \dots, \cdot]$ . Then

$$\begin{aligned} \rho[x_1, \dots, x_n]_{\rho} &= \rho(\rho[x_1, \dots, x_n]) \\ &= \rho[\rho(x_1), \dots, \rho(x_n)] \\ &= [\rho(x_1), \dots, \rho(x_n)]_{\rho}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & [[x_1, \dots, x_n]_{\rho}, \rho(y_1), \dots, \rho(y_{n-1})]_{\rho} \\ &= \rho[[x_1, \dots, x_n]_{\rho}, \rho(y_1), \dots, \rho(y_{n-1})] \\ &= \rho[\rho[x_1, \dots, x_{n-1}], \rho(y_1), \dots, \rho(y_{n-1})] \\ &= \rho^2[[x_1, \dots, x_n], y_1, \dots, y_{n-1}] \\ &= \rho^2 \sum_{i=1}^n [x_1, \dots, [x_i, y_1, \dots, y_{n-1}], \dots, x_n] \\ &= \sum_{i=1}^n [\rho(x_1), \dots, [x_i, y_1, \dots, y_{n-1}]_{\rho}, \dots, \rho(x_n)]_{\rho}. \end{aligned}$$

Therefore,  $(\mathfrak{g}, \rho \circ [\cdot, \dots, \cdot], \rho)$  is a multiplicative  $n$ -ary Hom-Nambu algebra.  $\square$

Similarly, we have

**Example 2.7.** Let  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$  be a multiplicative  $n$ -ary Hom-Nambu algebra and let  $\beta : \mathfrak{g} \rightarrow \mathfrak{g}$  be an  $n$ -ary Hom-Nambu algebra morphism. Then  $(\mathfrak{g}, \beta \circ [\cdot, \dots, \cdot], \beta \circ \alpha)$  is a multiplicative  $n$ -ary Hom-Nambu algebra.

### 3 Derivations of $n$ -ary Hom-Nambu algebra

Let  $(\mathfrak{g}, [\cdot, \dots, \cdot]_g, \alpha)$  be a multiplicative  $n$ -ary Hom-Nambu algebra. For any nonnegative integer  $k$ , denote by  $\alpha^k$  the  $k$ -times composition of  $\alpha$ , i.e.

$$\alpha^k = \alpha \circ \dots \circ \alpha \quad (k - \text{times}).$$

In particular,  $\alpha^0 = \text{Id}$  and  $\alpha^1 = \alpha$ .

**Definition 3.1.** For any nonnegative integer  $k$ , a linear map  $D : \mathfrak{g} \rightarrow \mathfrak{g}$  is called an  $\alpha^k$ -derivation of the multiplicative  $n$ -ary Hom-Nambu algebra  $(\mathfrak{g}, [\cdot, \dots, \cdot]_g, \alpha)$ , if

$$[D, \alpha] = 0, \quad \text{i.e.} \quad D \circ \alpha = \alpha \circ D,$$

and

$$D[x_1, \dots, x_n]_g = \sum_{i=1}^n [\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)]_g.$$

Denote by  $\text{Der}_{\alpha^s}(\mathfrak{g})$  is the set of  $\alpha^s$ -derivations of the multiplicative  $n$ -ary Hom-Nambu algebra  $(\mathfrak{g}, [\cdot, \dots, \cdot]_g, \alpha)$ . For any  $D \in \text{Der}_{\alpha^k}(\mathfrak{g})$  and  $D' \in \text{Der}_{\alpha^s}(\mathfrak{g})$ , define their commutator  $[D, D']$  as usual:

$$[D, D'] = D \circ D' - D' \circ D.$$

**Lemma 3.2.** For any  $D \in \text{Der}_{\alpha^k}(\mathfrak{g})$  and  $D' \in \text{Der}_{\alpha^s}(\mathfrak{g})$ , we have

$$[D, D'] \in \text{Der}_{\alpha^{k+s}}(\mathfrak{g}).$$

Denote by

$$\text{Der}(\mathfrak{g}) = \bigoplus_{k \geq 0} \text{Der}_{\alpha^k}(\mathfrak{g}).$$

By Lemma 3.2, obviously we have

**Proposition 3.3.** With the above notations,  $(\text{Der}(\mathfrak{g}), [\cdot, \cdot], \alpha')$  is a Hom-Lie algebra, with  $\alpha'(D) = D \circ \alpha$ .

At the end of this section, we consider the derivation extension of the multiplicative Hom-Leibniz algebra  $(\mathfrak{g}, [\cdot, \cdot]_g, \alpha)$  and give an application of the  $\alpha$ -derivation  $\text{Der}_{\alpha}(\mathfrak{g})$ .

For any linear map  $D : \mathfrak{g} \rightarrow \mathfrak{g}$ , consider the vector space  $\mathfrak{g} \oplus RD$ . Define a multilinear bracket operation  $[\cdot, \cdot]_g$  on  $\mathfrak{g} \oplus RD$  by

$$[x_1 + mD, x_2 + nD]_D = [x_1, x_2]_g + mD(x_2) - nD(x_1) \quad \forall x_1, x_2 \in \mathfrak{g}.$$

Define a linear map  $\alpha' : \mathfrak{g} \oplus RD \rightarrow \mathfrak{g} \oplus RD$  by  $\alpha'(x_1 + mD) = \alpha(x_1) + mD$ , i.e.

$$\alpha' = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

**Proposition 3.4.** *With the above notations, if  $(\mathfrak{g} \oplus RD, [\cdot, \cdot]_D, \alpha')$  is a multiplicative Hom-Leibniz algebra, then  $D$  is an  $\alpha$ -derivation of the multiplicative Hom-Leibniz algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha)$ .*

*Proof.* Since  $(\mathfrak{g} \oplus RD, [\cdot, \cdot]_D, \alpha')$  is a multiplicative Hom-Leibniz algebra, so we have

$$[[x, y]_D, \alpha'(D)]_D = [\alpha'(x), [y, D]_D]_D + [[x, D]_D, \alpha'(y)]_D,$$

that is  $D([x, y]) = [\alpha(x), D(y)] + [D(x), \alpha(y)]$ .  $\square$

For  $\mathcal{X} \in \mathfrak{g}^{\wedge^{n-1}}$  satisfying  $\alpha(\mathcal{X}) = \mathcal{X}$  and  $k \geq 0$ , we define the map  $\text{ad}_k(\mathcal{X}) \in \text{End}(\mathfrak{g})$  by

$$\text{ad}_k(\mathcal{X})(y) = [\alpha^k(y), x_1, \dots, x_{n-1}], \forall y \in \mathfrak{g}.$$

**Lemma 3.5.** *The map  $\text{ad}_k(\mathcal{X})$  is an  $\alpha^{k+1}$ -derivation and is called an inner  $\alpha^{k+1}$ -derivation.*

*Proof.* For any  $y_1, \dots, y_n \in \mathfrak{g}$ , we have

$$\begin{aligned} & \text{ad}_{k-1}(\mathcal{X})[y_1, \dots, y_n] \\ &= [\alpha^k[y_1, \dots, y_n], x_1, \dots, x_{n-1}] \\ &= [[\alpha^k(y_1), \dots, \alpha^k(y_n)], \alpha(x_1), \dots, \alpha(x_{n-1})] \\ &= \sum_{i=1}^n [\alpha^{k+1}(y_1), \dots, \alpha^{k+1}(y_{i-1}), [\alpha^k(y_i), \alpha(x_1), \dots, \alpha(x_{n-1})], \dots, \alpha^{k+1}(y_n)] \\ &= \sum_{i=1}^n [\alpha^{k+1}(y_1), \dots, \alpha^{k+1}(y_{i-1}), \text{ad}_{k-1}(\mathcal{X})(y_i), \dots, \alpha^{k+1}(y_n)]. \end{aligned}$$

Then  $\text{ad}_{k-1}(\mathcal{X}) \in \text{Der}_{\alpha^{k+1}}(\mathfrak{g})$ .  $\square$

We denote by  $\text{Inn}_{\alpha^k}(\mathfrak{g})$  the  $\mathbb{K}$ -vector space generated by all inner  $\alpha^{k+1}$ -derivations.

**Proposition 3.6.** *The  $\text{Inn}_{\alpha^k}(\mathfrak{g})$  is a Hom-ideal of  $\text{Der}(\mathfrak{g})$ .*

*Proof.*  $(\text{Der}(\mathfrak{g}), [\cdot, \cdot], \alpha')$  is a Hom-Lie algebra. We show that  $\text{Inn}(\mathfrak{g})$  is a Hom-ideal. Let  $\text{ad}_{k-1}(\mathcal{X})(y) = [\alpha^{k-1}(y), x_1, \dots, x_{n-1}]$  be an inner  $\alpha^k$ -derivation on  $\mathfrak{g}$  and  $D \in \text{Der}_{\alpha^{k'}}(\mathfrak{g})$  for  $k \geq 1$  and  $k' \geq 0$ . Then  $[D, \text{ad}_{k-1}(\mathcal{X})] \in \text{Der}_{\alpha^{k+k'}}(\mathfrak{g})$  and for any  $y \in \mathfrak{g}$

$$\begin{aligned} & [D, \text{ad}_{k-1}(\mathcal{X})](y) \\ &= D[\alpha^{k-1}(y), x_1, \dots, x_{n-1}] - [\alpha^{k-1}(D(y)), x_1, \dots, x_{n-1}] \\ &= [D\alpha^{k-1}(y), \alpha^{k'}(x_1), \dots, \alpha^{k'}(x_{n-1})] \\ & \quad + \sum_{i=1}^{n-1} [\alpha^{k+k'-1}(y), \alpha^{k'}(x_1), \dots, D(x_i), \dots, \alpha^{k'}(x_{n-1})] \\ & \quad - [\alpha^{k-1}(Dy), \alpha^{k'}(x_1), \dots, \alpha^{k'}(x_{n-1})] \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{n-1} [\alpha^{k+k'-1}(y), \alpha^{k'}(x_1), \dots, D(x_i), \dots, \alpha^{k'}(x_{n-1})] \\
&= \sum_{i=1}^{n-1} [\alpha^{k+k'-1}(y), x_1, \dots, D(x_i), \dots, x_{n-1}] \\
&= \sum_{i=1}^{n-1} \text{ad}_{k+k'-1}(x_1 \wedge \dots \wedge D(x_i) \wedge \dots \wedge x_{n-1})(y).
\end{aligned}$$

Therefore,  $[D, \text{ad}_{k-1}(\mathcal{X})] \in \text{Inn}_{\alpha^{k+k'-1}}(\mathfrak{g})$ . And we have

$$\alpha'(\text{ad}_{k-1}(\mathcal{X}))(y) = \text{ad}_{k-1}(\mathcal{X}) \circ \alpha(y) = \text{ad}_k(\mathcal{X})(y).$$

So the  $\text{Inn}_{\alpha^k}(\mathfrak{g})$  is a Hom-ideal of  $\text{Der}(\mathfrak{g})$ . □

## 4 $\omega$ -Derivations

**Definition 4.1.** Let  $\mathcal{A}$  be a vector space equipped with an  $n$ -linear map  $\omega : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ ,  $\alpha$  is a linear map on  $\mathcal{A}$ . A map  $f : \mathcal{A} \rightarrow \mathcal{A}$  is called an  $\omega$ - $\alpha^k$ -derivation if it satisfies

$$f(\omega(a_1, \dots, a_n)) = \sum_{i=1}^n \omega(\alpha^k(a_1), \dots, f(a_i), \dots, \alpha^k(a_n));$$

$$f \circ \alpha = \alpha \circ f.$$

Denote by  $\text{Der}_{\alpha^k}^{\omega}(\mathcal{A})$  the set of  $\omega$ - $\alpha^k$ -derivations of the vector space  $\mathcal{A}$ . For any  $f \in \text{Der}_{\alpha^k}^{\omega}(\mathcal{A})$  and  $g \in \text{Der}_{\alpha^s}^{\omega}(\mathcal{A})$ , define their commutator  $[f, g]$  as usual:

$$[f, g] = f \circ g - g \circ f.$$

**Lemma 4.2.** For any  $f \in \text{Der}_{\alpha^k}^{\omega}(\mathcal{A})$  and  $g \in \text{Der}_{\alpha^s}^{\omega}(\mathcal{A})$ , we have

$$[f, g] \in \text{Der}_{\alpha^{k+s}}^{\omega}(\mathcal{A}).$$

Denote by

$$\text{Der}^{\omega}(\mathcal{A}) = \bigoplus_{k \geq 0} \text{Der}_{\alpha^k}^{\omega}(\mathcal{A}).$$

By Lemma 4.2, obviously we have

**Proposition 4.3.** With the above notations,  $(\text{Der}^{\omega}(\mathcal{A}), [\cdot, \cdot], \alpha')$  is a Hom-Lie algebra, with  $\alpha'(f) = f \circ \alpha$ .

**Lemma 4.4.** If  $f \in \text{Der}_{\alpha^k}^{\omega}(\mathcal{A})$  and  $f \in \text{Der}_{\alpha^k}^{\sigma}(\mathcal{A})$ , then  $f \in \text{Der}_{\alpha^k}^{\omega+\sigma}(\mathcal{A})$ . Here  $\sigma : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$  is also an  $n$ -linear map.

*Proof.* If  $f \in \text{Der}^\omega(\mathcal{A})$  and  $f \in \text{Der}^\sigma(\mathcal{A})$ , then we have

$$\begin{aligned}
& f((\omega + \delta)(a_1, \dots, a_n)) \\
&= f(\omega(a_1, \dots, a_n)) + f(\sigma(a_1, \dots, a_n)) \\
&= \sum_{i=1}^n \omega(\alpha^k(a_1), \dots, f(a_i), \dots, \alpha^k(a_n)) + \sum_{i=1}^n \sigma(\alpha^k(a_1), \dots, f(a_i), \dots, \alpha^k(a_n)) \\
&= \sum_{i=1}^n (\omega + \sigma)(\alpha^k(a_1), \dots, f(a_i), \dots, \alpha^k(a_n)).
\end{aligned}$$

So we get the conclusion.  $\square$

**Example 4.5.** Let  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$  be a multiplicative  $n$ -ary Hom-Nambu algebra. we define  $\omega : \mathfrak{g}^{\otimes n} \rightarrow \mathfrak{g}$

$$\omega(x_1, \dots, x_n) = [x_1, \dots, x_n]$$

for any  $x_1, \dots, x_n \in \mathfrak{g}$ . If  $f \in \text{Der}(\mathfrak{g})$ , then  $f \in \text{Der}^\omega(\mathfrak{g})$ .

**Proposition 4.6.** Let  $\omega_i : \mathcal{A}^{\otimes n_i} \rightarrow \mathcal{A}$  be  $n_i$ -linear maps satisfying  $\omega_i \circ \alpha = \alpha \circ \omega_i$  for  $i = 1, \dots, k$  and let  $\omega : \mathcal{A}^{\otimes k} \rightarrow \mathcal{A}$  be a  $k$ -linear map. If  $f \in \bigcap_{i=1}^k \text{Der}_{\alpha^t}^{\omega_i}(\mathcal{A}) \cap \text{Der}_{\alpha^t}^\omega(\mathcal{A})$ , then  $f \in \text{Der}_{\alpha^t}^\sigma(\mathcal{A})$  with  $\sigma : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ . Here  $n = n_1 + \dots + n_k$ ,  $\sigma(a_1, \dots, a_n) \triangleq \omega(\omega_1(a_1, \dots, a_{n_1}), \dots, \omega_k(a_1, \dots, a_{n_k}))$  and  $s = n - n_k + 1 = n_1 + \dots + n_{k-1} + 1$ .

*Proof.* On the one hand, we have

$$\begin{aligned}
& f(\sigma(a_1, \dots, a_n)) \\
&= f(\omega(\omega_1(a_1, \dots, a_{n_1}), \dots, \omega_k(a_s, \dots, a_n))) \\
&= \sum_{i=1}^k \omega(\alpha^t(\omega_1(a_1, \dots, a_{n_1})), \dots, f(\omega_i(a_{m_1}, \dots, a_{m_{n_i}})), \dots, \alpha^t(\omega_k(a_s, \dots, a_n))) \\
&= \sum_{i=1}^k \omega(\omega_1(\alpha^t(a_1), \dots, \alpha^t(a_{n_1}))), \dots, f(\omega_i(a_{m_1}, \dots, a_{m_{n_i}})), \dots, \omega_k(\alpha^t(a_s), \dots, \alpha^t(a_n))),
\end{aligned}$$

and here

$$f(\omega_i(a_{m_1}, \dots, a_{m_{n_i}})) = \sum_{j=1}^{m_{n_i}} \omega_i(\alpha^t(a_{m_1}), \dots, f(a_{m_{n_j}}), \dots, \alpha^t(a_{m_{n_i}})).$$

On the other hand,

$$\begin{aligned}
& \sum_{i=1}^n \sigma(\alpha^t(a_1), \dots, f(a_i), \dots, \alpha^t(a_n)) \\
&= \sum_{i=1}^n \omega(\omega_1(\alpha^t(a_1), \dots, \alpha^t(a_{n_1}))), \dots, \omega_i(\alpha^t(a_{p_1}), \dots, f(a_{p_t}), \dots, \omega_k(\alpha^t(a_s), \dots, \alpha^t(a_n))).
\end{aligned}$$

So we have  $f(\sigma(a_1, \dots, a_n)) = \sum_{i=1}^n \sigma(\alpha^t(a_1), \dots, f(a_i), \dots, \alpha^t(a_n))$ .  $\square$

**Lemma 4.7.** Let  $\omega : \mathcal{A}^{\otimes(n+1)} \rightarrow \mathcal{A}$  be an  $(n+1)$ -linear map and let  $\mu_i : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  be bilinear maps given by

$$\mu_i(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n) = \alpha^k(a_1) \otimes \cdots \otimes \omega(a_i, b_1, \dots, b_n) \otimes \cdots \otimes \alpha^k(a_n).$$

Here  $\mathfrak{g} = \mathcal{A}^{\otimes n}$  and  $1 \leq i \leq n$ . Suppose that  $f \in \text{Der}_{\alpha^0}^\omega(\mathcal{A})$  and  $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$  is given by

$$\varphi(a_1 \otimes \cdots \otimes a_n) = \sum_{j=1}^n a_1 \otimes \cdots \otimes f(a_j) \otimes \cdots \otimes a_n.$$

Then  $\varphi \in \text{Der}_{\alpha^0}^{\mu_i}(\mathcal{A})$ .

*Proof.* On the one hand, we have

$$\begin{aligned} & \varphi(\mu_i(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n)) \\ &= \varphi(\alpha^k(a_1) \otimes \cdots \otimes \omega(a_i, b_1, \dots, b_n) \otimes \cdots \otimes \alpha^k(a_n)) \\ &= \sum_{j=1}^{i-1} \alpha^k(a_1) \otimes \cdots \otimes f\alpha^k(a_j) \otimes \cdots \otimes \omega(a_i, b_1, \dots, b_n) \otimes \cdots \otimes \alpha^k(a_n) \\ &+ \sum_{j=i+1}^n \alpha^k(a_1) \otimes \cdots \otimes \omega(a_i, b_1, \dots, b_n) \otimes \cdots \otimes f\alpha^k(a_j) \otimes \cdots \otimes \alpha^k(a_n) \\ &+ \alpha^k(a_1) \otimes \cdots \otimes f(\omega(a_i, b_1, \dots, b_n)) \otimes \cdots \otimes \alpha^k(a_n) \\ &= \sum_{j=1}^{i-1} \alpha^k(a_1) \otimes \cdots \otimes \alpha^k f(a_j) \otimes \cdots \otimes \omega(a_i, b_1, \dots, b_n) \otimes \cdots \otimes \alpha^k(a_n) \\ &+ \sum_{j=i+1}^n \alpha^k(a_1) \otimes \cdots \otimes \omega(a_i, b_1, \dots, b_n) \otimes \cdots \otimes \alpha^k f(a_j) \otimes \cdots \otimes \alpha^k(a_n) \\ &+ \sum_{j=1}^n \alpha^k(a_1) \otimes \cdots \otimes \omega(a_i, b_1, \dots, f(b_j), \dots, b_n) \otimes \cdots \otimes \alpha^k(a_n) \\ &+ \alpha^k(a_1) \otimes \cdots \otimes \omega(f(a_i), b_1, \dots, b_n) \otimes \cdots \otimes \alpha^k(a_n). \end{aligned}$$



The other hand,

$$\begin{aligned}
& \mu_i(\varphi(a_1 \otimes \cdots \otimes a_n), b_1 \otimes \cdots \otimes b_n) + \mu_i(a_1 \otimes \cdots \otimes a_n, \varphi(b_1 \otimes \cdots \otimes b_n)) \\
&= \mu_i\left(\sum_{j=1}^n a_1 \otimes \cdots \otimes f(a_j) \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n\right) \\
&+ \mu_i\left(a_1 \otimes \cdots \otimes a_n, \sum_{j=1}^n b_1 \otimes \cdots \otimes f(b_j) \otimes \cdots \otimes b_n\right) \\
&= \sum_{j=1}^n \mu_i(a_1 \otimes \cdots \otimes f(a_j) \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n) \\
&+ \sum_{j=1}^n \mu_i(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes f(b_j) \otimes \cdots \otimes b_n) \\
&= \sum_{j=1}^{i-1} \alpha^k(a_1) \otimes \cdots \otimes \alpha^k f(a_j) \otimes \cdots \otimes \omega(a_i, b_1, \dots, b_n) \otimes \cdots \otimes \alpha^k(a_n) \\
&+ \sum_{j=i+1}^n \alpha^k(a_1) \otimes \cdots \otimes \omega(a_i, b_1, \dots, b_n) \otimes \cdots \otimes \alpha^k f(a_j) \otimes \cdots \otimes \alpha^k(a_n) \\
&+ \alpha^k(a_1) \otimes \cdots \otimes \omega(f(a_i), b_1, \dots, b_n) \otimes \cdots \otimes \alpha^k(a_n) \\
&+ \sum_{j=1}^n \alpha^k(a_1) \otimes \cdots \otimes \omega(a_i, b_1, \dots, f(b_j), \dots, b_n) \otimes \cdots \otimes \alpha^k(a_n).
\end{aligned}$$

So we have

$$\begin{aligned}
& \varphi(\mu_i(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n)) \\
&= \mu_i(\varphi(a_1 \otimes \cdots \otimes a_n), b_1 \otimes \cdots \otimes b_n) + \mu_i(a_1 \otimes \cdots \otimes a_n, \varphi(b_1 \otimes \cdots \otimes b_n)),
\end{aligned}$$

that is  $\varphi \in \text{Der}_{\alpha^0}^{\mu_i}(\mathcal{A})$ . □

**Theorem 4.8.** (1) Let  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$  be a multiplicative  $(n+1)$ -ary Hom-Nambu algebra. Then  $D_n(\mathfrak{g})$  is a Leibniz algebra with respect to the bracket

$$[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n] \triangleq \sum_{i=1}^n a_1 \otimes \cdots \otimes [a_i, b_1, \dots, b_n] \otimes \cdots \otimes a_n.$$

(2) Moreover, if we define

$$[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n] \triangleq \sum_{i=1}^n \alpha(a_1) \otimes \cdots \otimes [a_i, b_1, \dots, b_n] \otimes \cdots \otimes \alpha(a_n)$$

and  $\alpha' : D_n(\mathfrak{g}) \rightarrow D_n(\mathfrak{g})$  satisfying

$$\alpha'(a_1 \otimes \cdots \otimes a_n) = \alpha(a_1) \otimes \cdots \otimes \alpha(a_n).$$

Then  $(D_n(\mathfrak{g}), [\cdot, \cdot], \alpha')$  is a multiplicative Hom-Leibniz algebra.

*Proof.* (1) From Lemma 4.7, let  $k = 1$ , and

$$\mu_i(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n) = [a_1, \cdots, [a_i, b_1, \cdots, b_n], \cdots, a_n].$$

Fix  $x_1, \cdots, x_n \in \mathfrak{g}$ , define  $f(a) = [a, x_1, \cdots, x_n]$ ,  $\varphi : D_n(\mathfrak{g}) \rightarrow D_n(\mathfrak{g})$  is given by

$$\varphi(a_1 \otimes \cdots \otimes a_n) = \sum_{j=1}^n a_1 \otimes \cdots \otimes f(a_j) \otimes \cdots \otimes a_n.$$

From Lemma 4.7, we get  $\varphi \in \text{Der}_{\alpha^0}^{\mu_i}(\mathfrak{g})$ . And from Lemma 4.4, we know  $\varphi \in \text{Der}_{\alpha^0}^{\mu}(\mathfrak{g})$ . Here  $\mu = \mu_1 + \cdots + \mu_n$ , and

$$\mu(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n) = [a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n].$$

$\varphi \in \text{Der}_{\alpha^0}^{\mu}(\mathfrak{g})$ , that is

$$\begin{aligned} \varphi(\mu(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n)) &= \mu(\varphi(a_1 \otimes \cdots \otimes a_n), b_1 \otimes \cdots \otimes b_n) \\ &+ \mu(a_1 \otimes \cdots \otimes a_n, \varphi(b_1 \otimes \cdots \otimes b_n)), \end{aligned}$$

that is

$$\begin{aligned} \varphi([a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n]) &= [\varphi(a_1 \otimes \cdots \otimes a_n), b_1 \otimes \cdots \otimes b_n] \\ &+ [a_1 \otimes \cdots \otimes a_n, \varphi(b_1 \otimes \cdots \otimes b_n)]. \end{aligned}$$

It is equivalent to

$$\begin{aligned} &[[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n], c_1 \otimes \cdots \otimes c_n] \\ &= [[a_1 \otimes \cdots \otimes a_n, c_1 \otimes \cdots \otimes c_n], b_1 \otimes \cdots \otimes b_n] \\ &+ [a_1 \otimes \cdots \otimes a_n, [b_1 \otimes \cdots \otimes b_n, c_1 \otimes \cdots \otimes c_n]]. \end{aligned}$$

So  $D_n(\mathfrak{g})$  is a Leibniz algebra.

(2) According to the definition of bracket, we have

$$\begin{aligned} &[[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n], \alpha(c_1) \otimes \cdots \otimes \alpha(c_n)] \\ &= \left[ \sum_{i=1}^n \alpha(a_i) \otimes \cdots \otimes [a_i, b_1 \otimes \cdots \otimes b_n] \otimes \cdots \otimes \alpha(a_n), \alpha(c_1) \otimes \cdots \otimes \alpha(c_n) \right] \\ &= \sum_{i=1}^n [\alpha(a_i) \otimes \cdots \otimes [a_i, b_1 \otimes \cdots \otimes b_n] \otimes \cdots \otimes \alpha(a_n), \alpha(c_1) \otimes \cdots \otimes \alpha(c_n)] \\ &= \sum_{i=1}^n \sum_{j=1}^{i-1} \alpha^2(a_1) \otimes \cdots \otimes [\alpha(a_j), \alpha(c_1), \cdots, \alpha(c_n)] \otimes \cdots \otimes \alpha([a_i, b_1, \\ &\quad \cdots, b_n]) \otimes \cdots \otimes \alpha^2(a_n) \\ &+ \sum_{i=1}^n \sum_{j=i+1}^n \alpha^2(a_1) \otimes \cdots \otimes \alpha([a_i, b_1, \cdots, b_n]) \otimes \cdots \otimes [\alpha(a_j), \alpha(c_1), \end{aligned} \tag{1}$$

$$\cdots, \alpha(c_n)] \otimes \cdots \otimes \alpha^2(a_n) \quad (2)$$

$$+ \sum_{i=1}^n \alpha^2(a_1) \otimes \cdots \otimes [[a_i, b_1 \otimes \cdots \otimes b_n], \alpha(c_1), \cdots, \alpha(c_n)] \otimes \cdots \otimes \alpha^2(a_n). \quad (3)$$

In the same way,

$$\begin{aligned} & [[a_1 \otimes \cdots \otimes a_n, c_1 \otimes \cdots \otimes c_n], \alpha(b_1) \otimes \cdots \otimes \alpha(b_n)] \\ &= \sum_{i=1}^n \sum_{j=1}^{i-1} \alpha^2(a_1) \otimes \cdots \otimes [\alpha(a_j), \alpha(b_1), \cdots, \alpha(b_n)] \otimes \cdots \otimes \alpha([a_i, c_1 \otimes \cdots \otimes c_n]) \otimes \\ & \quad \cdots \otimes \alpha^2(a_n) \end{aligned} \quad (2')$$

$$\begin{aligned} &+ \sum_{i=1}^n \sum_{j=i+1}^n \alpha^2(a_1) \otimes \cdots \otimes \alpha([a_i, c_1 \otimes \cdots \otimes c_n]) \otimes \cdots \otimes [\alpha(a_j), \alpha(b_1), \cdots, \alpha(b_n)] \otimes \\ & \quad \cdots \otimes \alpha^2(a_n) \end{aligned} \quad (1')$$

$$+ \sum_{i=1}^n \alpha^2(a_1) \otimes \cdots \otimes [[a_i, c_1 \otimes \cdots \otimes c_n], \alpha(b_1), \cdots, \alpha(b_n)] \otimes \cdots \otimes \alpha^2(a_n). \quad (3')$$

And

$$\begin{aligned} & [\alpha(a_1) \otimes \cdots \otimes \alpha(a_n), [b_1 \otimes \cdots \otimes b_n, c_1 \otimes \cdots \otimes c_n]] \\ &= \sum_{i=1}^n [\alpha(a_1) \otimes \cdots \otimes \alpha(a_n), \alpha(b_1) \otimes \cdots \otimes [b_i, c_1, \cdots, c_n] \otimes \cdots \otimes \alpha(b_n)] \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha^2(a_1) \otimes \cdots \otimes [\alpha(a_j), \alpha(b_1), \cdots, [b_i, c_1, \cdots, c_n], \cdots, \alpha(b_n)] \otimes \cdots \otimes \alpha^2(a_n). \end{aligned} \quad (3'')$$

Obviously,  $(1) = (1'), (2) = (2'), (3) = (3') + (3'')$ , so

$$\begin{aligned} & [[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n], \alpha(c_1) \otimes \cdots \otimes \alpha(c_n)] \\ &= [[a_1 \otimes \cdots \otimes a_n, c_1 \otimes \cdots \otimes c_n], \alpha(b_1) \otimes \cdots \otimes \alpha(b_n)] \\ &+ [\alpha(a_1) \otimes \cdots \otimes \alpha(a_n), [b_1 \otimes \cdots \otimes b_n, c_1 \otimes \cdots \otimes c_n]]. \end{aligned}$$

So  $(D_n(\mathfrak{g}), [\cdot, \cdot], \alpha')$  is a multiplicative Hom-Leibniz algebra.  $\square$

Moreover, we can prove

**Corollary 4.9.** *If  $\mathfrak{g}$  is a  $(kn+1)$ -ary Hom-Nambu algebra, then  $(\mathfrak{g}^{\otimes k}, [\cdot, \cdots, \cdot], \alpha')$  is a Hom-Leibniz  $(n+1)$ -algebra with respect to the following bracket*

$$\begin{aligned} & [x_{01} \otimes \cdots \otimes x_{0k}, \cdots, x_{n1} \otimes \cdots \otimes x_{nk}] \\ & \triangleq [x_{01}, \cdots, x_{11}, \cdots, x_{1k}, \cdots, x_{n1}, \cdots, x_{nk}] \otimes \alpha(x_{02}) \otimes \cdots \otimes \alpha(x_{0k}) \\ & + \cdots + \alpha(x_{01}) \otimes \cdots \otimes \alpha(x_{0k-1}) \otimes [x_{0k}, x_{11}, \cdots, x_{nk}] \end{aligned}$$

and  $\alpha' : \mathfrak{g}^{\otimes k} \rightarrow \mathfrak{g}^{\otimes k}$  satisfying

$$\alpha'(a_1 \otimes \cdots \otimes a_k) = \alpha(a_1) \otimes \cdots \otimes \alpha(a_k).$$

## 5 Representations of $n$ -ary Hom-Nambu algebra

**Definition 5.1.** Let  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$  be an  $n$ -ary Hom-Nambu algebra. We say  $M$  is a representation of  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ , if  $M$  is a vector space with a multilinear map  $[\cdot, \dots, \cdot] : \mathfrak{g}^{\otimes i} \otimes M \otimes \mathfrak{g}^{\otimes n-1-i} \longrightarrow M$  satisfying

$$\begin{aligned} & [[x_1, \dots, x_n], \alpha'(y_1), \dots, \alpha'(y_{n-1})] \\ &= \sum_{i=1}^n [\alpha'(x_1), \dots, \alpha'(x_{i-1}), [x_i, y_1, \dots, y_{n-1}], \alpha'(x_{i+1}), \dots, \alpha'(x_n)], \end{aligned}$$

and one of  $x_1, \dots, x_n, y_1, \dots, y_n$  in  $M$ , others in  $\mathfrak{g}$ . Here  $\alpha' : M \oplus \mathfrak{g} \longrightarrow M \oplus \mathfrak{g}$  satisfying

$$\alpha'(x) = \alpha(x), \forall x \in \mathfrak{g};$$

$$\alpha'(m) = m, \forall m \in M.$$

**Proposition 5.2.** Let  $M$  be a representation of  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha')$ , and  $f : \mathfrak{g}^{\otimes n} \longrightarrow M$  is an  $n$ -linear map. We define an  $n$ -bracket on  $H = M \oplus \mathfrak{g}$  by

$$[(m_1, x_1), \dots, (m_n, x_n)] \triangleq \left( \sum_{i=1}^n [x_1, \dots, m_i, \dots, x_n] + f(x_1, \dots, x_n), [x_1, \dots, x_n] \right)$$

and  $\alpha' : M \oplus \mathfrak{g} \longrightarrow M \oplus \mathfrak{g}$  satisfying

$$\alpha'(m, x) = (\alpha(x), m), \forall x \in \mathfrak{g}, m \in M.$$

Then  $(H, [\cdot, \dots, \cdot], \alpha')$  is an  $n$ -ary Hom-Nambu algebra if and only if

$$\begin{aligned} & f([x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})) + [f(x_1, \dots, x_n), \alpha(y_1), \dots, \alpha(y_{n-1})] \\ &= \sum_{i=1}^n (f(\alpha(x_1), \dots, [x_i, y_1, \dots, y_{n-1}], \dots, \alpha(x_n)) \\ &+ [\alpha(x_1), \dots, f([x_i, y_1, \dots, y_{n-1}], \dots, \alpha(x_n))]). \end{aligned} \tag{5.1}$$

*Proof.* On the one hand, we have

$$\begin{aligned} & [[(m_1, x_1), \dots, (m_n, x_n)], \alpha'(p_1, y_1), \dots, \alpha'(p_{n-1}, y_{n-1})] \\ &= [(\sum_{i=1}^n [x_1, \dots, m_i, \dots, x_n] + f(x_1, \dots, x_n), [x_1, \dots, x_n]), \alpha'(p_1, y_1), \dots, \alpha'(p_{n-1}, y_{n-1})] \\ &= (\sum_{j=1}^{n-1} [[x_1, \dots, x_n], \dots, p_j, \dots, \alpha(y_{n-1})] + [\sum_{i=1}^n [x_1, \dots, m_i, \dots, x_n] + f(x_1, \dots, x_n), \\ &\quad \alpha(y_1), \dots, \alpha(y_{n-1})] + f([x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})), [[x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})]) \\ &= (\sum_{j=1}^{n-1} [[x_1, \dots, x_n], \dots, p_j, \dots, \alpha(y_{n-1})], \end{aligned} \tag{1}$$

$$[[x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})]] \quad (2)$$

$$+ \left( \sum_{i=1}^n [[x_1, \dots, m_i, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})], \right. \quad (3)$$

$$\begin{aligned} & [[x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})]] \\ & + ([f(x_1, \dots, x_n), \alpha(y_1), \dots, \alpha(y_{n-1})], [[x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})]]) \\ & + (f([x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})), [[x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})]]). \end{aligned}$$

The other hand,

$$\begin{aligned} & \sum_{i=1}^n [\alpha'(m_1, x_1), \dots, \alpha'(m_{i-1}, x_{i-1}), [(m_i, x_i), (p_1, y_1), \dots, (p_{n-1}, y_{n-1})], \dots, \alpha'(m_n, x_n)] \\ & = \sum_{i=1}^n [(m_1, \alpha(x_1)), \dots, (m_{i-1}, \alpha(x_{i-1})), [(m_i, x_i), (p_1, y_1), \dots, (p_{n-1}, y_{n-1})], \dots, (m_n, \alpha(x_n))] \\ & = \sum_{i=1}^n [(m_1, \alpha(x_1)), \dots, (m_{i-1}, \alpha(x_{i-1})), \left( \sum_{j=1}^{n-1} [x_i, \dots, p_j, \dots, y_{n-1}] + [m_i, y_1, \dots, y_{n-1}] + \right. \\ & \quad \left. f(x_i, y_1, \dots, y_{n-1}), [x_i, y_1, \dots, y_{n-1}] \right), \dots, (m_n, \alpha(x_n))] \\ & = \left( \sum_{i=1}^n \sum_{j=1}^{n-1} [\alpha(x_1), \dots, \alpha(x_{i-1}), [x_i, \dots, p_j, \dots, y_{n-1}], \dots, \alpha(x_n)], \right. \quad (1') \end{aligned}$$

$$\sum_{i=1}^n [\alpha(x_1), \dots, \alpha(x_{i-1}), [x_i, y_1, \dots, y_{n-1}], \alpha(x_{i+1}), \dots, \alpha(x_n)]) \quad (2')$$

$$+ \left( \sum_{i=1}^n [\alpha(x_1), \dots, \alpha(x_{i-1}), [m_i, \dots, y_1, \dots, y_{n-1}], \dots, \alpha(x_n)], \right. \quad (3')$$

$$\begin{aligned} & \sum_{i=1}^n [\alpha(x_1), \dots, \alpha(x_{i-1}), [x_i, y_1, \dots, y_{n-1}], \alpha(x_{i+1}), \dots, \alpha(x_n)]) \\ & + \left( \sum_{i=1}^n [\alpha(x_1), \dots, \alpha(x_{i-1}), f(x_i, y_1, \dots, y_{n-1}), \alpha(x_{i+1}), \dots, \alpha(x_n)], \right. \\ & \sum_{i=1}^n [\alpha(x_1), \dots, \alpha(x_{i-1}), [x_i, y_1, \dots, y_{n-1}], \alpha(x_{i+1}), \dots, \alpha(x_n)]) \\ & + \left( \sum_{i=1}^n f(\alpha(x_1), \dots, \alpha(x_{i-1}), [x_i, y_1, \dots, y_{n-1}], \alpha(x_{i+1}), \dots, \alpha(x_n)), \right. \\ & \sum_{i=1}^n [\alpha(x_1), \dots, \alpha(x_{i-1}), [x_i, y_1, \dots, y_{n-1}], \alpha(x_{i+1}), \dots, \alpha(x_n)]). \end{aligned}$$

Obviously, (1) = (1'), (2) = (2'), (3) = (3'), so  $(H, [\cdot, \dots, \cdot], \alpha')$  is an  $n$ -ary Hom-Nambu algebra if and only if

$$f([x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})) + [f(x_1, \dots, x_n), \alpha(y_1), \dots, \alpha(y_{n-1})]$$

$$\begin{aligned}
&= \sum_{i=1}^n (f(\alpha(x_1), \dots, [x_i, y_1, \dots, y_{n-1}], \dots, \alpha(x_n)) \\
&\quad + [\alpha(x_1), \dots, f([x_i, y_1, \dots, y_{n-1}], \dots, \alpha(x_n))].
\end{aligned}$$

Then we get the conclusion.  $\square$

Obviously, if the condition of the proposition 5.2 holds, then we obtain an extension

$$0 \rightarrow M \rightarrow H \rightarrow \mathfrak{g} \rightarrow 0$$

of  $n$ -ary Hom-Nambu algebras. Moreover,

**Proposition 5.3.** *This extension is split in the category of  $n$ -ary Hom-Nambu algebras if and only if there exists a linear map  $h : \mathfrak{g} \rightarrow M$  such that*

$$\mathfrak{g} \circ \alpha' = \alpha' \circ \mathfrak{g};$$

$$f(x_1, \dots, x_n) = \sum_{i=1}^n [x_1, \dots, x_n] - h([x_1, \dots, x_n]). \quad (5.2)$$

An easy consequence of these facts is the following natural bejection:

$$Ext(\mathfrak{g}, M) \cong Z(\mathfrak{g}, M)/B(\mathfrak{g}, M). \quad (5.3)$$

Here  $Ext(\mathfrak{g}, M)$  is the set of isomorphism classes of extensions of  $\mathfrak{g}$  by  $M$ ,  $Z(\mathfrak{g}, M)$  is the set of all linear maps  $f : \mathfrak{g}^{\otimes n} \rightarrow M$  satisfying (5.1), and  $B(\mathfrak{g}, M)$  is the set of such  $f$  which satisfies (5.2) for some  $k$ -linear map  $h : \mathfrak{g} \rightarrow M$ .

We just to prove  $B(\mathfrak{g}, M) \subseteq Z(\mathfrak{g}, M)$ . In fact, if  $f$  satisfies (5.2), then

$$\begin{aligned}
&f([x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})) + [f(x_1, \dots, x_n), \alpha(y_1), \dots, \alpha(y_{n-1})] \\
&= [h([x_1, \dots, x_n]), \alpha(y_1), \dots, \alpha(y_{n-1})] \\
&\quad + \sum_{i=1}^{n-1} [[x_1, \dots, x_n], \alpha(y_1), \dots, h(\alpha(y_i)), \dots, \alpha(y_{n-1})] - h([x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})) \\
&\quad + \sum_{i=1}^n [[x_1, \dots, h(x_i), \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})] - [h([x_1, \dots, x_n]), \alpha(y_1), \dots, \alpha(y_{n-1})] \\
&= \sum_{i=1}^{n-1} [[x_1, \dots, x_n], \alpha(y_1), \dots, h(\alpha(y_i)), \dots, \alpha(y_{n-1})] \quad (1) \\
&\quad - h([x_1, \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})) \quad (2) \\
&\quad + \sum_{i=1}^n [[x_1, \dots, h(x_i), \dots, x_n], \alpha(y_1), \dots, \alpha(y_{n-1})]. \quad (3)
\end{aligned}$$

And

$$\begin{aligned}
& \sum_{i=1}^n (f(\alpha(x_1), \dots, [x_i, y_1, \dots, y_{n-1}], \dots, \alpha(x_n)) \\
& + [\alpha(x_1), \dots, f([x_i, y_1, \dots, y_{n-1}]), \dots, \alpha(x_n)]) \\
& = \sum_{i=1}^n \sum_{j=1}^{i-1} [\alpha(x_1), \dots, h(\alpha(x_j)), \dots, [x_i, y_1, \dots, y_{n-1}], \dots, \alpha(x_n)] \quad (3')
\end{aligned}$$

$$+ \sum_{i=1}^n \sum_{j=i+1}^n [\alpha(x_1), \dots, [x_i, y_1, \dots, y_{n-1}], \dots, h(\alpha(x_j)), \dots, \alpha(x_n)] \quad (3'')$$

$$+ \sum_{i=1}^n [\alpha(x_1), \dots, h([x_i, y_1, \dots, y_{n-1}]), \dots, \alpha(x_n)] \quad (4)$$

$$- \sum_{i=1}^n h([\alpha(x_1), \dots, [x_i, y_1, \dots, y_{n-1}], \dots, \alpha(x_n)]) \quad (2')$$

$$+ \sum_{i=1}^n \sum_{j=1}^{n-1} [\alpha(x_1), \dots, \alpha(x_{i-1}), [x_i, \dots, h(y_j), \dots, y_{n-1}], \alpha(x_{i+1}), \dots, \alpha(x_n)] \quad (1')$$

$$+ \sum_{i=1}^n [\alpha(x_1), \dots, \alpha(x_{i-1}), [h(x_i), \dots, y_{n-1}], \alpha(x_{i+1}), \dots, \alpha(x_n)] \quad (3''')$$

$$- \sum_{i=1}^n [\alpha(x_1), \dots, \alpha(x_{i-1}), h([x_i, \dots, y_{n-1}]), \alpha(x_{i+1}), \dots, \alpha(x_n)]. \quad (4')$$

Since  $(1) = (1')$ ,  $(2) = (2')$ ,  $(3) = (3') + (3'') + (3''')$ ,  $(4) = -(4')$ , so  $f$  satisfies (5.1).

**Example 5.4.** Let  $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha')$  be an  $(n+1)$ -ary Hom-Nambu algebra,  $M$  is a representation of  $\mathfrak{g}$ .  $\text{Hom}(\mathfrak{g}, M)$  is the set of isomorphism of  $\mathfrak{g} \rightarrow M$ . One defines the maps

$$\begin{aligned}
[-, -] &: \text{Hom}(\mathfrak{g}, M) \otimes \rightarrow \text{Hom}(\mathfrak{g}, M) \\
[-, -] &: D_n(\mathfrak{g}) \otimes \text{Hom}(\mathfrak{g}, M) \rightarrow \text{Hom}(\mathfrak{g}, M)
\end{aligned}$$

by

$$\begin{aligned}
[f, x_1 \otimes \dots \otimes x_n](x) &= -[f(x), x_1 \otimes \dots \otimes x_n], \\
[x_1 \otimes \dots \otimes x_n, f](x) &= -[f(x), x_1 \otimes \dots \otimes x_n].
\end{aligned}$$

Then  $(\text{Hom}(\mathfrak{g}, M), [-, -])$  is a representation of  $D_n(\mathfrak{g})$ , here  $D_n(\mathfrak{g}) = \mathfrak{g}^{\otimes n}$ .

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